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CDD Zeroes in the Pomeranchukon Scattering Amplitude

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ABSTRACT

Using dispersion relations and the properties of Herglotz functions we study solutions of two-Pomeranchukon unitarity in the presence of CDD zeroes. We conclude that the best solution is that in which two CDD zeroes are present and the triple Pomeranchukon vertex vanishes linearly. This structure leads to the factorization of cross sections to $0\left[\frac{1}{(\ell n\, s)^2}\right]$.

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The most powerful way of solving Reggeon cut discontinuity formula for the Pomeranchukon[1-3] should be the field-theoretic approach of the Reggeon calculus.[4] This is because the infinite set of coupled discontinuity formulae for all the multi-Pomeranchukon cuts are treated simultaneously. In fact this is probably the only way of obtaining a strong-coupling solution[5,6] in which all the cuts contribute in a similar fashion to the total cross section. However, in the weak coupling case, where the cuts remain separable from the pole at $t = 0$, the critical problem is the collision of just the two Pomeranchukon cut and the pole which involves the three Pomeranchukon vertex. For this reason it is actually more straightforward to discuss the weak coupling case using S-Matrix methods.

A basic treatment of this problem was first given by Bronzan.[7] His approach is to study possible analytic forms for the Froissart-Gribov amplitude which contain a self-consistent Pomeranchukon pole and are consistent with both analyticity in t and the N/D form required by the two Pomeranchukon discontinuity formula. In this paper we use a different approach and obtain solutions which have a different zero structure to those given by Bronzan. Our approach is based on a combination of dispersion relations with the properties of Herglotz functions and is closer to that of Abarbanel,[3] except that we allow for the presence of "CDD" zeroes which are critical for obtaining a satisfactory solution. Our conclusion will be that the best solution is that in which two CDD zeroes are present. This requires the triple Pomeranchukon vertex to vanish linearly in the forward direction. This is the solution recently obtained by Cardy and White[8] in the context of the Reggeon calculus with an effective singular potential. In that context one CDD zero can be interpreted as the triple Pomeranchukon zero resulting from the singular potential, while the other is interpreted as the "bare" Pomeranchukon pole trajectory before its interaction with the two Pomeranchukon cut.

We consider therefore the t-channel "on-shell" Pomeranchukon-Pomeranchukon partial-wave amplitude $A\left\{j, t; t_1 = t_2, \alpha(t_1) + \alpha(t_2) = j+1\right\} \equiv A(j,t)$ (see Fig. 1 for notation). We shall ignore the structure of this amplitude in the t_1 and t_2-channels and in the t-channel consider only the Pomeranchukon pole at $j = \alpha(t)$ and the associated two Pomeranchukon cut at $j = \alpha_c(t) = 2\alpha(\frac{t}{4}) - 1$. We assume that we can write a dispersion relation for $A(j,t)$ in j for fixed positive t, in the form

$$A(j,t) = \frac{[i\, g(t)]^2}{j - \alpha(t)} + \frac{1}{\pi}\int_{-\infty}^{\alpha_c(t)} \frac{dj'}{(j'-j)} \operatorname{Im} A(j',t) + C \qquad (1)$$

where the residue of the pole is the square of the full triple-Pomeranchukon vertex which we shall assume to be pure imaginary.* C is a subtraction constant which we shall initially take to be zero but will later find to be essential. Note that the positive sign for the cut contribution in (1) corresponds to a <u>negative</u> contribution of this cut to the total cross

*This is a well known result of the Reggeon calculus and is a consequence of the requirement that the two Pomeranchukon cut be negative in lowest order Reggeon perturbation theory.

section.[9] The two Pomeranchukon discontinuity formula requires that $A(j,t)$ satisfies the unitarity relation

$$\mathrm{Im}\, A(j,t) = \rho(j,t)\left| A(j,t)\right|^2 \tag{2}$$

where the kinematical factor $\rho(j,t)$ is essentially given by

$$\rho(j,t) = \beta / \left(\frac{d\alpha}{dt'}\right)_{t' = t_0(j)} \tag{3}$$

$t_0(j)$ being the solution of $j = 2\alpha(t_0) - 1$ and β is a function of j, which can be treated as a positive constant.

It follows from (1) that $\mathrm{Im}\, A(j,t) \geq 0$ for $\mathrm{Im}\, j > 0$ and so $A(j,t)$ and therefore $-A^{-1}(j,t)$ are Herglotz-type functions.[10, 11] Accordingly the solution of (1) takes the form

$$-A^{-1}(j,t) = a + bj + \frac{j}{\pi}\int_{-\infty}^{\alpha_c(t)} \frac{dj'\ \rho(j',t)}{j'(j'-j)} + \sum_i \frac{R_i(t)}{j_i(t) - j} \tag{4}$$

with the properties that

$$b = \lim_{j\to\infty} - \frac{A^{-1}(j,t)}{j} \geq 0. \tag{5}$$

and

$$R_i(t) \geq 0, \quad t > 0 \tag{6}$$

To see whether "CDD poles" at $j = j_i(t)$ should be present, consider $A(j,t)$ in the region of j between the branch point $\alpha_c(t)$ and the pole position $\alpha(t)$. From (1), we see that the pole contribution is always real and positive while the second term representing the cut contribution is real and negative. Therefore, if $C = 0$ and the cut contribution is sufficiently strong to make $A(j,t)$ negative near the branch point $\alpha_c(t)$ then there must be a zero, say, at $j = j_1(t)$, somewhere between $\alpha_c(t)$ and $\alpha(t)$. A plot of $A(j,t)$ against j, in this case, is shown in Fig. 2.

By normalizing (4) so that $A(j,t)$ has a pole at $j = \alpha(t)$ with the residue $[ig(t)]^2$ we can eliminate a and b and write

$$-A^{-1}(j,t) = \frac{j-\alpha(t)}{g^2(t)} + \frac{[j-\alpha(t)]^2}{\pi}\int_{-\infty}^{\alpha_c(t)} \frac{dj'\,\rho(j',t)}{(j'-j)[j'-\alpha(t)]^2} + \frac{R_1(t)\,[j-\alpha(t)]^2}{[j_1(t)-j]\,[j_1(t)-\alpha]^2} \tag{7}$$

where we have kept the nearby zero term explicitly. In particular the residue condition yields

$$b = \frac{1}{g^2(t)} - \int_{-\infty}^{\alpha_c(t)} dj'\,\frac{\rho(j',t)}{[j'-\alpha(t)]^2} - \frac{R_1}{[j_1(t)-\alpha(t)]^2} \tag{8}$$

the right-hand side of which is positive definite from the Herglotz property (5) so that we obtain the inequality relation

$$g^2(t) \le \left[\frac{1}{\pi}\int_{-\infty}^{\alpha_c(t)} dj\,\frac{\rho(j,t)}{[j-\alpha(t)]^2} + \frac{R_1(t)}{[j_1(t)-\alpha(t)]^2}\right]^{-1} \tag{9}$$

The inequality (9) ensures that the solution given by (9) has no other poles beyond the one at $j = \alpha(t)$. The equality holds when $A(j,t)$ decreases less fast than $\frac{1}{j}$ asymptotically.

Since $j_1(t)$ lies between $\alpha(t)$ and $\alpha_c(t)$ it follows that $j_1(0) = \alpha(0) = \alpha_c(0) = 1$. If $j_1(t)$ is analytic at $t = 0$ [as well as $\alpha(t)$] so that

$j_1(t) = 1 + j_1'(0)t + \ldots$ and if $R_1(0)$ is finite then we can deduce from (9) that

$$g(t) \underset{t\to 0}{<} 0\,(t) \tag{10}$$

so that g(t) vanishes linearly at t = 0. This result is to be contrasted with Abarbanel's result,[3] which follows from (9) in the absence of the CDD zero term, that

$$g(t) \underset{t\to 0}{<} 0\,[(\alpha-\alpha_c)^{\frac{1}{2}}] = 0(\sqrt{t}\,) \tag{11}$$

Unfortunately, the simple solution we have obtained with a single CDD zero is not acceptable as we now show. By taking $\alpha(t) = 1 + \alpha' t$ so that $\rho(j,t) = \beta/2\alpha'$ from (3), Eq. (7) can be simply integrated to give

$$-A^{-1}(j,t) = \frac{j-\alpha(t)}{g^2(t)}\left\{1 + \frac{\beta g^2(t)}{2\alpha'\pi}\left[\frac{1}{\alpha_c-\alpha} + \frac{1}{(j-\alpha)}\,\ell n\left(\frac{j-\alpha_c}{\alpha-\alpha_c}\right)\right]\right.$$
$$\left. + \frac{g^2(t)R_1(t)(j-\alpha)}{[j_1(t)-j\,]\,[j_1(t)-\alpha(t)]^2}\right\} \tag{12}$$

We can now see that our solution differs from Abarbanel's[3] only by the addition of the CDD pole term. It also suffers from the same disease as Abarbanel's solution in that the presence of $\ell n(\alpha-\alpha_c)$ in (12) leads to a fixed t-singularity of A(j,t) at t = 0. This we know cannot be allowed because of the analyticity properties of the Froissart-Gribov representation.

If the logarithmic singularity is to be cancelled in (12) it is clear that the CDD term must be involved because of the j-dependence of the

the coefficient of $\ln(\alpha-\alpha_c)$. To see whether a cancellation is possible we write

$$\frac{R_1(j-\alpha)}{(j_1-j)(j_1-\alpha)^2} = R_1\left[\frac{1}{(j_1-j)(j_1-\alpha)} - \frac{1}{(j_1-\alpha)^2}\right] \tag{13}$$

Since $j_1 - \alpha < 0$, $R_1 > 0$ and $j_1 \to \alpha$ it is clear that the logarithmic can be cancelled, so that $A(j,t)$ is analytic at $t = 0$ if

$$\frac{R_1}{(j_1-\alpha)} \sim -\ln(\alpha-\alpha_c), \quad \frac{R_1}{(j_1-\alpha)^2} \sim \frac{1}{t^n} \tag{14}$$

$$n = 1, 2, \ldots$$

which requires that

$$j_1(t) - \alpha(t) \underset{t\to 0}{\sim} -t^n \ln(\alpha-\alpha_c) \tag{15}$$

and

$$R_1(t) \underset{t\to 0}{\sim} t^n \left[\ln(\alpha-\alpha_c)\right]^2 \tag{16}$$

To avoid too dramatic behavior for R_1 we take $n = 1$. If we now return to Eq. (9), we see that we can no longer deduce (10) but instead obtain only (11). However, if $g(t) \underset{t\to 0}{\sim} \sqrt{t}$ we encounter another difficulty. The two-particle/two Pomeranchukon amplitude $N(j,t)$ (the familiar fixed-pole residue) is obtained from $A(j,t)$ by[1, 2, 7]

$$N(j,t) = N_0(j,t) \times A(j,t) \tag{17}$$

where $N_0(j,t)$ is regular at $j = \alpha_c(t)$. $N_0(j,t)$ enables us to introduce a finite two particle/Pomeranchukon vertex function so that the Pomeranchukon can give a finite contribution to the four-particle amplitude $f(j,t)$ through the relation[1,2,7]

$$f(j,t) = N_0^2 A + E(j,t) \tag{18}$$

$E(j,t)$ also being regular at $j = \alpha_c$. We therefore have to take

$$N_0(\alpha,t) \underset{t\to 0}{\sim} [g(t)]^{-1} \sim t^{-\frac{1}{2}} \tag{19}$$

and this introduces an illegal singularity at $t=0$ in $N(j,t)$. Rather than resort to $n=2$ in (14) we go on to consider the possibility of further CDD zeroes in $A(j,t)$.

If $C=0$ in (1) then it is clear from Fig. 2 that there cannot be more than one CDD zero. However, if $C > 0$ then the situation shown in Fig. 3 is possible. Now a second CDD zero, to the right of $\alpha(t)$, is possible (note that $\mathrm{Im}\, A \neq 0$ for $j < \alpha_c$ and so a real zero to the left of α_c is not possible). In this case (9) becomes

$$g^2(t) = \left[\frac{1}{\pi}\int_{-\infty}^{\alpha_c(t)} dj \frac{\rho(j,t)}{[j-\alpha(t)]^2} + \frac{R_1}{(j_1-\alpha)^2} + \frac{R_2}{(j_2-\alpha)^2}\right]^{-1} \tag{20}$$

Now we are free to choose R_1 and j_1 to satisfy (14) with $n=1$ and still let $j_2(t)$ be analytic at $t=0$ with $j_2(0) = 1$ so that

$$j_2 - \alpha \underset{t\to 0}{\sim} t \tag{21}$$

and (10) still holds. We now have that

$$-g^2(t)A^{-1}(j,t) \underset{t\to 0}{\sim} [j-\alpha(t)]\left\{1 + [\frac{g^2 R_2}{(j_2-\alpha)^2}]\right\} \tag{22}$$

and since the four-particle amplitude $f(j,t)$ satisfies (18) we have

$$f(j,t) \underset{t \to 0}{\sim} g^{-2}(t) A(j,t) \tag{23}$$

and it is clear that $f(j,t)$ still contains a pole at $t=0$.

We conclude therefore that a satisfactory partial-wave amplitude can be constructed containing a Pomeranchukon pole, a two-Pomeranchukon cut and two CDD zeroes. It is interesting to note that this is precisely the structure found by Cardy and White[8] on the basis of a Reggeon calculus model. In their notation

$$P_0 \equiv j - j_2 \qquad Z_0 \equiv j - j_1 \tag{24}$$

so that P_0 contains the "bare" Pomeranchukon trajectory, while Z_0 results directly from the singular potential. The reconstruction of the two particle/two Pomeranchukon and four particle amplitudes from the four-Pomeranchukon amplitude can be followed through in an analagous way to that given in Ref. 8, leading to the important conclusion that a zero structure of the sort we have considered leads directly to the factorization of total cross sections to $0\left[\frac{1}{(\ln s)^2}\right]$.

To see how this factorization comes about and also to compare with Bronzan's solutions we briefly recall the analysis of Ref. 8. The crucial part of the analysis is the introduction of the "one-Pomeranchukon irreducible" amplitude A_I which does not contain the Pomeranchukon pole but satisfies the unitarity relation (2). It follows then that A_I can be written in the form

$$A_I = \frac{1}{\tilde{A} - \frac{\beta}{2\pi\alpha'} \ln(j-\alpha_c)} \tag{25}$$

where $\tilde{A}$ is regular at $j = \alpha_c$.

Then

$$A(j,t) = \frac{1}{\tilde{A} - \tilde{D}^2/P_0 - \frac{\beta}{2\pi\alpha'} \ln(j-\alpha_c)} \tag{26}$$

where $\tilde{D}$ is regular at $j = \alpha_c$ and is essentially the three Pomeranchukon vertex before two Pomeranchukon iterations are considered. The four-particle amplitude $f_{ab}(j,t)$ for the scattering of two particles a and b then takes the form

$$f_{ab}(j,t) = \tilde{E} + \frac{\tilde{g}_a\tilde{g}_b}{P_0} + \frac{(\tilde{c}_a + \tilde{g}_a\tilde{D}/P_0)(\tilde{c}_b + \tilde{g}_b\tilde{D}/P_0)}{\tilde{A} - \tilde{D}^2/P_0 - \frac{\beta}{2\pi\alpha'} \ln(j-\alpha_c)} \tag{27}$$

where $\tilde{E}$ does not contain P_0 and is regular at $j = \alpha_c$. $\tilde{g}_{a,b}$ and $\tilde{c}_{a,b}$ are, respectively, two-particle/Pomeranchukon and two particle/two Pomeranchukon vertex functions, before two-Pomeranchukon iteration. $\tilde{c}_a$ and $\tilde{c}_b$ are what in general prevents the two-Pomeranchukon cut contribution in (27) from factorizing.

It follows immediately from (26) that $A(j,t)$ has one CDD zero at $P_0 = 0$. Since P_0 contains the Pomeranchukon trajectory function before iterations of the two Pomeranchukon cut are added it <u>must</u> be analytic at $t = 0$ and therefore <u>has</u> to be our second CDD zero $j_2(t)$. If (26) contains a second CDD zero whose trajectory function is singular at $t = 0$, as required by (15), then it must correspond to a pole of $\tilde{A}$

rather than of $\tilde{D}^2$. This is what was attributed to the singular nature of the two Pomeranchukon interaction potential in Ref.8. If $\tilde{D} \not\to 0$ when $t \to 0$, it follows from (25) that

$$f_{ab}(j,0) \sim \tilde{g}_a \tilde{g}_b \left[\frac{1}{(P_0 - \tilde{D}^2/\tilde{A})} + \frac{\tilde{D}^2 \beta / 2\pi\alpha'}{(P_0 \tilde{A} - \tilde{D}^2)} \ln(j - \alpha_c) + \ldots \right] \tag{28}$$

so that $\tilde{c}_a$ and $\tilde{c}_b$ do not contribute to the leading contribution of the two Pomeranchukon cut at $t = 0$ and factorization holds. If $\tilde{D} \underset{t \to 0}{\to} 0$ then P_0 will not play the role required of $j_2(t)$ in (20), (21), and (10).

It seems therefore that (28) will follow directly from the CDD zero structure we have argued for. The above argument is, of course, based on the existence of the one-Pomeranchukon irreducible amplitude A_I. This is automatic in the Reggeon calculus but may have a more general basis.

It is interesting to note that Bronzan[7] argued against single order zeroes in $A(j,t)$ essentially on the basis that such zeroes would have to be cancelled in $f_{ab}(j,t)$ (to give a finite Pomeranchukon pole residue) by single order poles in N_0^2, that is the numerator of the last term in (25). He argued that because of (17) this would lead to illegal square root branch points in the two particle/two Pomeranchukon amplitude. However, (25) illustrates how a single order zero (P_0) can be allowed. There is a second order pole in the relevant numerator and the resulting pole is explicitly cancelled by the addition of the term $g_a g_b / P_0$. Second order zeroes of the type considered by Bronzan destroy the

Herglotz property we have based our analysis on. This is closely linked with the presence of further "ghost" poles in $A(j,t)$ besides that at $j = \alpha_p(t)$. While there is nothing in principle against introducing such poles (provided they lie to the left of $j = 1$ at $t = 0$) our use of the Herglotz property does enable us to avoid them in a simple way. Finally we note that our discussion of the cancellation of the logarithm in (12) using (14) is incomplete in the sense that we have only discussed leading order effects. We have not discussed (as Bronzan does) the complete elimination of any singularity at $t = 0$, with a completely self-consistent form for $\alpha_p(t)$. It seems likely that this can be done by an iteration process similar to that used by Bronzan. However, it also seems likely that we cannot really avoid the weak fixed cut at $j=1$ which his arguments imply will emerge from this process. We have tacitly assumed we can ignore this effect in writing equation (1).

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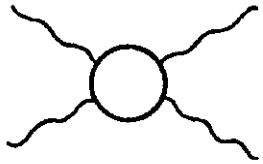

Fig. 1

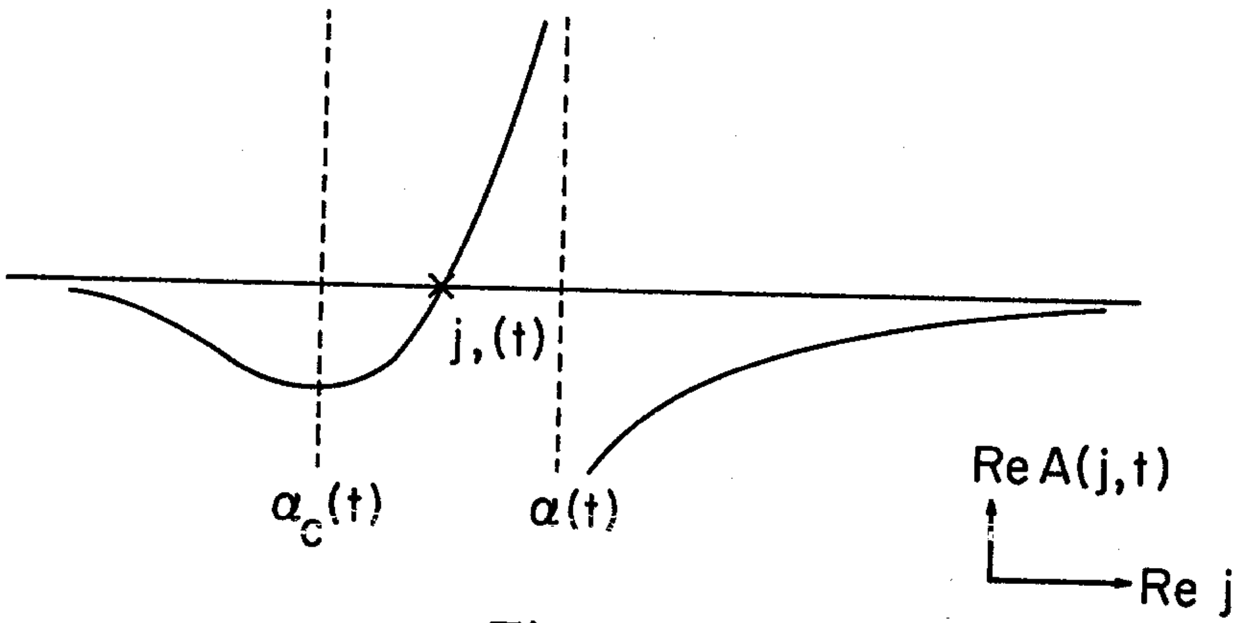

Fig. 2

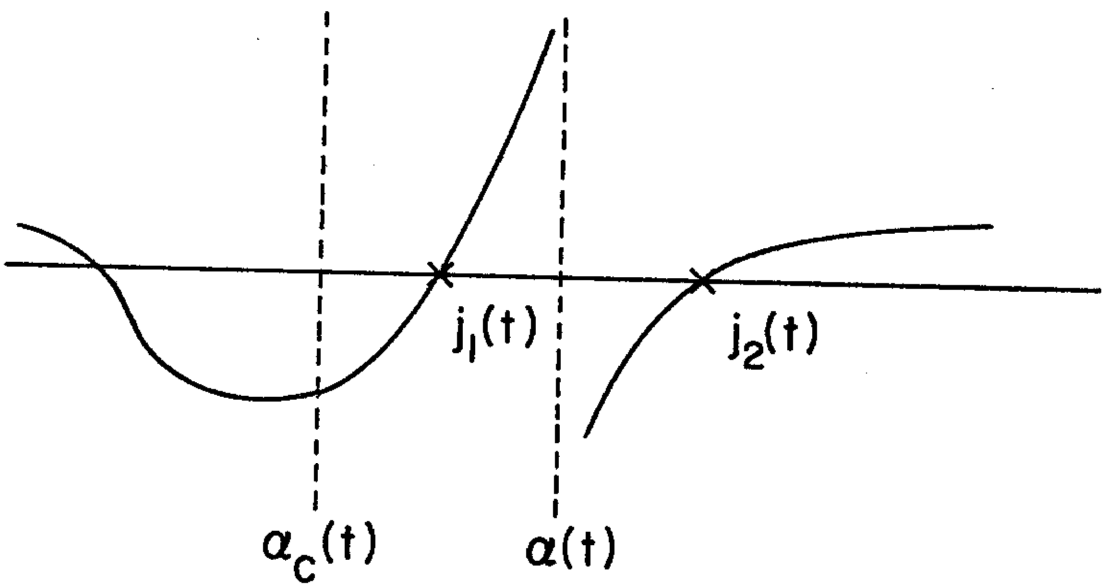

Fig. 3